SYNCHRONIZATION REGIMES IN SYSTEMS WITH MULTIPLE TIME-DELAYS

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ABSTRACT

We present the first report on chaos synchronization between two unidirectionally linearly and nonlinearly coupled systems with multiple time-delays and using the Razumikhin-Lyapunov approach find the existence and stability conditions for different synchronization regimes. The approach is tested on the famous nonlinear models-Ikeda, Mackey-Glass and Lang-Kobayashi systems.
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Introduction.- There is continued growth in the field of chaos control [1] and ever-increasing appreciation of its appications among researchers. Application of chaos control can be found in secure communication, optimization of nonlinear system performance, modeling brain activity and pattern recognition phenomena [2], species population control [3],etc.

Recently delay differential equations (DDEs) have attracted a lot of attention in the field of nonlinear dynamics. DDEs are used to model dynamical systems in many scientific and enginering areas,e.g.optics, biology, climatology, economy, cryptosystems based on synchronized hyperchaos, networks [4-5], etc. In comparison with a single time-delay DDEs with multiple time-delays are more relaistic models in the interacting complex systems. These delays result naturally from the finite propagation velocity of information, from the latency of feedback loops, from the finite switching times between different states of the system. Additional time-delays could be useful e.g. to stabilize nonlinear system's output [6].The application possibilities based on chaos require proper control of complexity. To the best our knowledge chaos synchronization between the systems with multiple time-delays has not been investigated yet. Having in mind enormous application implications of chaos synchronization e.g. in secure communication, performance optimization in nonlinear systems, stabilization problems, etc. investigation of synchronization regimes in the multiple time-delay systems is of paramount importance.

In this Letter we present the first report on chaos synchronization between two unidirectionally linearly and nonlinearly coupled chaotic systems with mutiple time-delays and find the existence and stability conditions for different synchronization regimes. We test the approach on the paradigm Ikeda, Mackey-Glass and Lang-Kobayashi models [4]. We hope this research will pave the way for the intensive experimental investigations of chaos synchronization in the multple time-delays systems.

General approach.-Consider synchronization between the double-feedback systems of general

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form ,

$$\frac{dx}{dt} = -\alpha x + m_1 f(x_{\tau_1}) + m_2 f(x_{\tau_2}), \tag{1}$$

$$\frac{dy}{dt} = -\alpha y + m_3 f(y_{\tau_1}) + m_4 f(y_{\tau_2}) + K f(x_{\tau_3}), \tag{2}$$

where f is differentiable generic nonlinear function. Throughout this paper $x_\tau \equiv x(t-\tau)$.
One finds that under the condition

$$m_1 - K = m_3, m_2 = m_4 \tag{3}$$

Eqs. (1) and (2) admit the synchronization manifold

$$y = x_{\tau_3-\tau_1}. \tag{4}$$

This follows from the dynamics of the error $\Delta = x_{\tau_3-\tau_1} - y$

$$\frac{d\Delta}{dt} = -\alpha\Delta + m_3\Delta_{\tau_1} f'(x_{\tau_3}) + m_2\Delta_{\tau_2} f'(x_{\tau_2+\tau_3-\tau_1}). \tag{5}$$

Here f' stands for the derivative of f with respect to time and the derivative should be bounded. The sufficient stability condition of the trivial solition $\Delta = 0$ of (5) can be found using Razumikhin theorems (see for details,[5],pp.151-161). In [5] considering the Lyapunov function $V = \frac{1}{2}\Delta^2$ and using Razumikhin theorems (hereafter we use the term Razumikhin-Lyapunov approach), under restrictive inital data condition $|\Delta(t)| \geq |\Delta(t+\theta)|$, for all $\theta \in [-\tau_j, 0]$ it is shown that zero solition of $\frac{d\Delta}{dt} = -a(t)\Delta(t) - \sum_{j=1}^{j=n} b_j(t)\Delta(t-\tau_j(t))$ is uniformly asymptotically stable for all bounded continous functions a, b_j, τ_j if $a(t) \geq \delta > 0, \sum_{j=1}^{j=n} b_j(t) < p\delta, 0 < p < 1, 0 < \tau_j(t) < \tau$ for all $t \in (-\infty, \infty)$. Thus, applying the Razumikhin-Lyapunov approach we obtain that the sufficient stability condition for the synchronization manifold (4) can be written as:

$$\alpha > |m_3(\sup f'(x_{\tau_3}))| + |m_2(\sup f'(x_{\tau_2+\tau_3-\tau_1}))|. \tag{6}$$

Here $\sup f'(x)$ stands for the supremum of f' with respect to the appropriate norm.
We notice that for $\tau_3 > \tau_1$,$\tau_3 = \tau_1$, and $\tau_3 < \tau_1$ (4) is the retarded, complete, and anticipating synchronization manifold [7,8], respectively.Analogously one finds both the existence ($m_2 - K = m_4$, $m_1 = m_3$) and sufficient stability ($\alpha > |m_3(\sup f'(x_{\tau_3}))| + |m_2(\sup f'(x_{\tau_2+\tau_3-\tau_1}))|$) conditions for synchronization manifold $y = x_{\tau_3-\tau_2}$.
In the case of linear coupling of form $K(x-y)$ one obtains that under the condition $m_1 = m_3, m_2 = m_4$ synchronization manifold $y = x$ exists and it is stable if $\alpha + K > |m_1(\sup f'(x_{\tau_1}))| + |m_2(\sup f'(x_{\tau_2}))|$. For the parameter mismatches, e.g. $\tau_1 \neq \tau_{1f}$, (τ_{1f} is the delay time for the first

feedback loop in y dynamics) it is clear that complete synchronization is not the synchronization manifold. Then for such a case we use the auxiliary system method to detect generalized synchronization [9]:that is given another identical driven auxiliary system $z(t)$, generalized synchronization between $x(t)$ and $y(t)$ is established with the achievement of complete synchronization between $y(t)$ and $z(t)$. Thus,the auxiliary method allows to find the local stability condition of the generalized synchronization [9]. Applying this method we find local stability condition of the generalized synchronization between y and x: $\alpha + K > |m_3(\sup f'(y_{\tau_{1f}}))| + |m_4(\sup f'(y_{\tau_2}))|$. We also notice for Kx_{τ_3} type of coupling with mismatches between the relaxation coefficents the only possible synchronization manifold is $y = x_{\tau_3}$ (independent of the relation between the coupling and feedback delay times) with existence $\alpha_2 - \alpha_1 = K$,$m_1 = m_3$, and $m_2 = m_4$ and sufficent stability conditions $\alpha_2 > |m_1(\sup f'(x_{\tau_1+\tau_3}))| + |m_2(\sup f'(x_{\tau_2+\tau_3}))|$. The presence of such a mismatch could be useful in the interpretation of the future experiments with coupling-delay lag synchronization and could serve as a "switch off" mechanizm for certain types of synchronization manifolds.
Generalization of the approach to n-tuple feedback systems, i.e. systems with multiple delays of type (1) and (2) is straightforward. We underline that a stability condition derived from the Razumikhin-Lyapunov approach is a sufficient condition: it assures a high quality synchronization for a coupling strength estimated from the stability condition, but does not forbid the possibility of synchronization with smaller coupling strengths. The threshold coupling strength can be estimated by the dependence of the maximal Lyapunov exponent λ of the error dynamics on K:i.e. from $\lambda(K) = 0$ [10].

Example 1:The Ikeda model.- First we test the approach on the nonlinearly coupled Ikeda model: $\frac{dx}{dt} = -\alpha x + m_1 \sin x_{\tau_1} + m_2 \sin x_{\tau_2}$; $\frac{dy}{dt} = -\alpha y + m_3 \sin y_{\tau_1} + m_4 \sin y_{\tau_2} + K \sin x_{\tau_3}$, with positive $\alpha_{1,2}$ and $-m_{1,2,3,4}$.This investigation is of considerable practical importance, as the equations of the class B lasers with feedback (typical representatives of class B are solid-state, semiconductor, and low pressure CO_2 lasers [11]) can be reduced to an equation of the Ikeda type [12].The Ikeda model was introduced to describe the dynamics of an optical bistable resonator, plays an important role in electronics and physiological studies and is well-known for delay-induced chaotic behavior [7,13]. We find that the Ikeda systems can be synchronized on the synchronization manifold $y = x_{\tau_3 - \tau_1}$ under the condition $m_1 - K = m_3, m_2 = m_4$ and is stable if $\alpha > |m_3| + |m_2|$. Numerical simulations fully support the analytical results. The Ikeda model was simulated using the DDE23 program [14] in MATLAB 6. Figure 1 shows the time series of the driver $x(t)$ (solid line) and the driven system $y(t)$(dotted line) for $\alpha = 5$, $\tau_1 = 3$,$\tau_2 = 2$,$\tau_3 = 1$, $m_2 = m_4 = -1$,$m_1 = -18$, $m_3 = -1$ and $K = -17$. After transients the driven system shifted $\tau_3 - \tau_1 = -2$ time units to the left and $y = x(t + 2)$ (anticipating synchronization).

Example 2: The Mackey-Glass model.-Consider complete synchronization between the linearly coupled double-feedback Mackey-Glass systems: $\frac{dx}{dt} = -\alpha_1 x + k_1 \frac{x_{\tau_1}}{1+x^b_{\tau_1}} + k_2 \frac{x_{\tau_2}}{1+x^b_{\tau_2}}$; $\frac{dy}{dt} = -\alpha_2 y + k_3 \frac{y_{\tau_1}}{1+y^b_{\tau_1}} + k_4 \frac{y_{\tau_2}}{1+y^b_{\tau_2}} + K(x - y)$.The dynamical variable in the Mackey-Glass model is the concentration of the mature cells in blood at time t and the delay time is the time between the

initiation of cellular production in the bone marrow and the release of mature cells into the blood [1,10].(At present there is also an electronic analog of the Mackey-Glass system [10].) We find that the Mackey-Glass systems can be synchronized on the synchronization manifold $y = x$ under the existence $k_1 = k_3, k_2 = k_4$ and stability $\alpha + K > (k_1 + k_2)\frac{(b-1)^2}{4b}$ conditions. For analytical estimations we took into account that the absolute maximum of the function $|f'(x_\tau)|$ is obtained at $x_\tau = (\frac{b+1}{b-1})^{\frac{1}{b}}$ and is equal to $\frac{(b-1)^2}{4b}$ [10]. Figure 2 shows numerical simulation of the linearly coupled Mackey-Glass models:time series of the driver $x(t)$ (solid line) and driven system $y(t)$ (dotted line). The parameters are $\tau_1 = 14, \tau_2 = 20, \alpha = 1, b = 10, k_1 = k_3 = 2, k_2 = k_4 = 0.2, K = 5$. After transients the driven systems trajectory completely coincides with that of the driver system. In Figure 3 generalized synchronization between the linearly coupled Mackey-Glass models is shown for $\tau_1 = 14, \tau_{1f} = 16$;the other parameters are as in Fig.2.
We emphasize that as the coupling strength estimated from the stability condition gives a high-quality synchronization, the synchronization manifold is robust against perturbations of the coupling strength. As mentioned above the onset of synchronization occurs at the coupling stength when the maximal Lyapunov exponent of the error dynamics vanishes as function of K [10]. Our estimations show that for the parameters values as in Fig.2 the threshold value of K is:$K \approx -1.11$, which is far less than $K = 3.44$ found from the sufficient stability condition.

Example 3:The Lang-Kobayashi model.-As the last example we study synchronization between the unidirectionally coupled double-feedback Lang-Kobayashi systems [1,6-8] (laser diodes with a double external cavities, see [6] and references there-in): $\frac{dE_{1,2}}{dt} = \frac{(1+\iota\alpha_{1,2})}{2}(G_{1,2}(N_{1,2} - N_{01,02}) - \gamma_{1,2})E_{1,2} + k_{1,2}E_{1,2}(t-\tau_1)\exp(-\iota\omega\tau_1) + k_{3,4}E_{1,2}(t-\tau_2)\exp(-\iota\omega\tau_2) + k_5E_1(t-\tau_3)\exp(-\iota\omega\tau_3)$; $\frac{dN_{1,2}}{dt} = J_{1,2} - \gamma_{e1,e2}N_{1,2} - G_{1,2}(N_{1,2} - N_{01,02})|E_{1,2}|^2$, where $E_{1,2}$ are slowly varying complex fields for the master and slave lasers, respectively $N_{1,2}$ are the carrier densities;$N_{01,02}$ are the carrier densities at transparency;$\gamma_{1,2}$ are the cavity losses; $\alpha_{1,2}$ are the linewidth enhancement factors; $G_{1,2}$ are the optical gains; $k_{1,2}$ and $k_{3,4}$ are the feedback levels for the master and slave lasers, respectively.k_5 is the coupling rate;ω is the optical feedback frequency without feedback; $\tau_{1,2}$ are the round-trip times in the external cavities for the coupled lasers; τ_3 is the time flight between the master laser and slave laser (coupling delay time);$J_{1,2}$ are the injection currents; $\gamma_{e1,e2}^{-1}$ are the carrier lifetimes.The term k_5 exists only for the slave laser. In order to find possible synchronization regimes we compare e.g. equations for the dynamics of E_2 and N_2 with dynamics of $E_{1,\tau_3-\tau_1}$ and $N_{1,\tau_3-\tau_1}$ and find that for the case of identical lasers (all parameters are the same except for the feedback and coupling rates) Lang-Kobayashi systems can be synchronized on the synchronization manifold $I_2 = I_{1,\tau_3-\tau_1}$ if

$$k_1 = k_3 + k_5, k_2 = k_4 \tag{7}$$

as the intensities I_2 and $I_{1,\tau_3-\tau_1}$ ($I = |E|^2$) can be made identical under these conditions. We have found the same existence conditions for the unidirectionally coupled laser diodes with incoherent optical feedbacks [15]. Unfortunately the stability of the synchronization manifolds for the Lang-Kobayashi model practically can not be studied analytically, and therefore one have to rely on the

numerical methods. Figure 4 shows synchronization manifold I_2 vs.$I_{1,\tau_3-\tau_1}$ for parameters values $N_0 = 1.7\times10^8$,$G = 2.14\times10^4$, $\tau_1 = 10^{-8}s$,$\tau_2 = 1.5\times10^{-8}s$,$\tau_3 = 2\times10^{-8}$, $\alpha = 5$, $\gamma_e = 0.9\times10^{-9}s$, $\frac{2\pi}{\omega} = 635nm$,$J = 0.02\gamma N_0$, $k_1 = 10ns^{-1}$,$k_3 = 1ns^{-1}$,$k_5 = 9ns^{-1}$,$k_2 = k_4 = 100ns^{-1}$.

Conclusions.- By using the Razumikhin-Lyapunov approach we have presented the first report on different synchronization regimes between two unidirectionally coupled (linearly and nonlinearly) chaotic systems with multiple delays. We have successfully applied the approach to the paradigm models in nonlinear physics-the Ikeda, Mackey-Glass and Lang-Kobayashi models. We have found analytically the existence and whenever possible stability conditions for the anticpating,lag, complete and generalized synchronization regimes. We hope that this research opens up possiblities for highly anticipated intensive experimental investigations of chaos synchronization in systems with multiple time-delays.

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Figure captions

FIG.1.Numerical simulation of the Ikeda model: the time series of the driver $x(t)$ (solid line) and the driven system $y(t)$(dotted line);$y = x(t + \tau_3 - \tau_1) = x(t + 2)$. Dimensionless units.

FIG.2.Numerical simulation of the Mackey-Glass model: complete synchronization between y and x.Dimensionless units.

FIG.3.Numerical simulation of the Mackey-Glass model: generalized synchronization between y and x.Dimensionless units.

FIG.4. Numerical simulation of the Lang-Kobayashi model:the dependence of I_2 on $I_{1,\tau_3-\tau_1}$.

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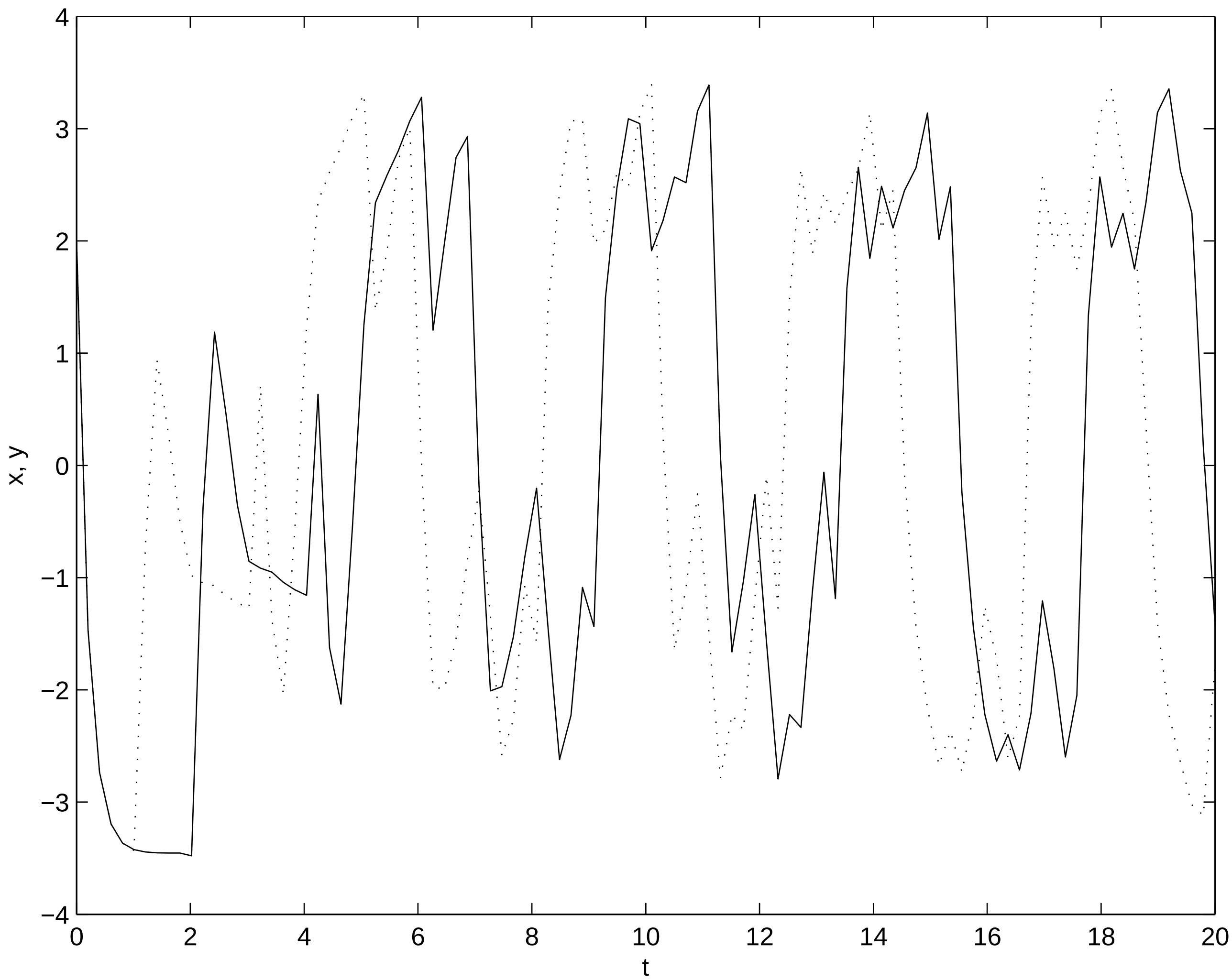
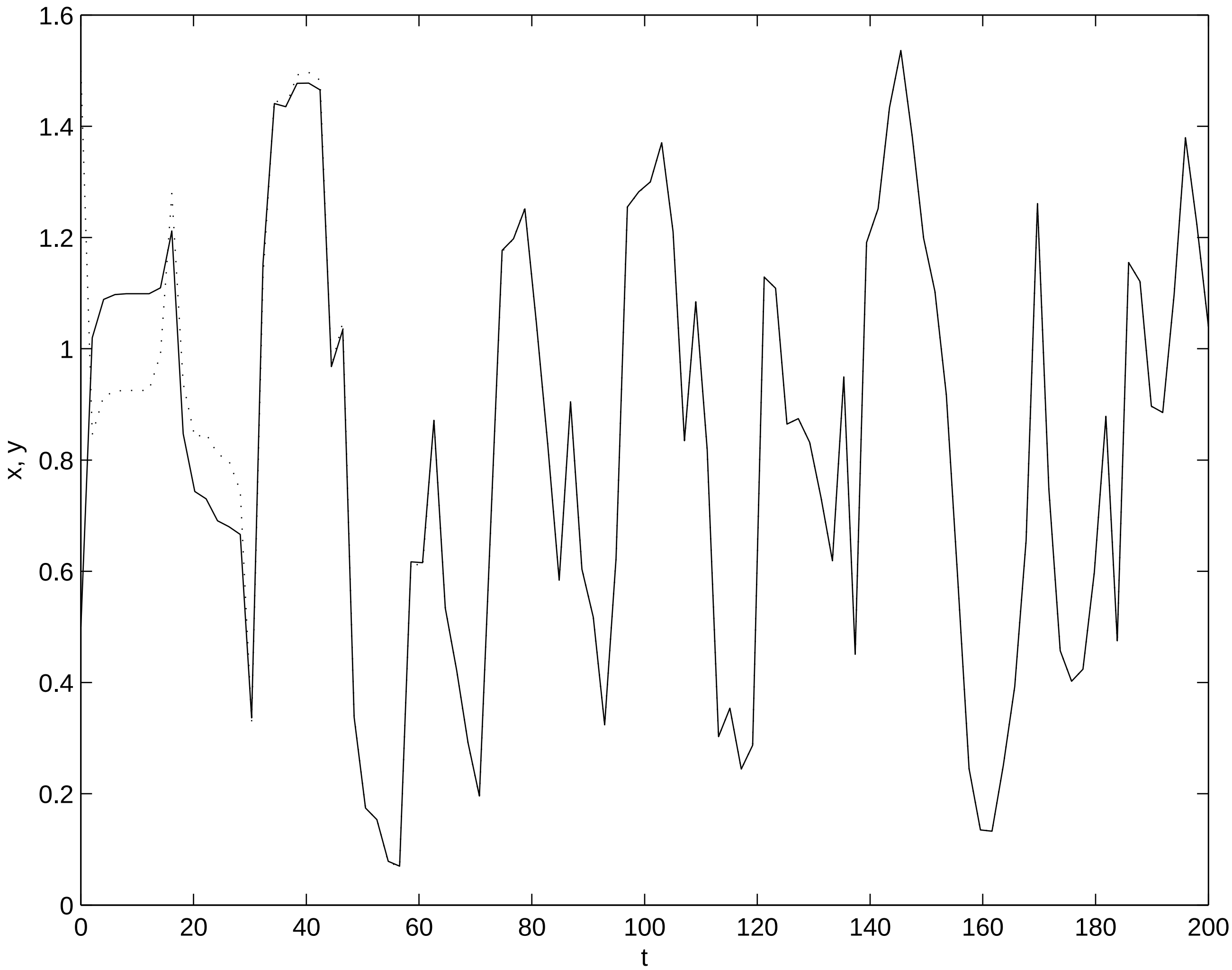


Fig.1.Shahverdiev PRL



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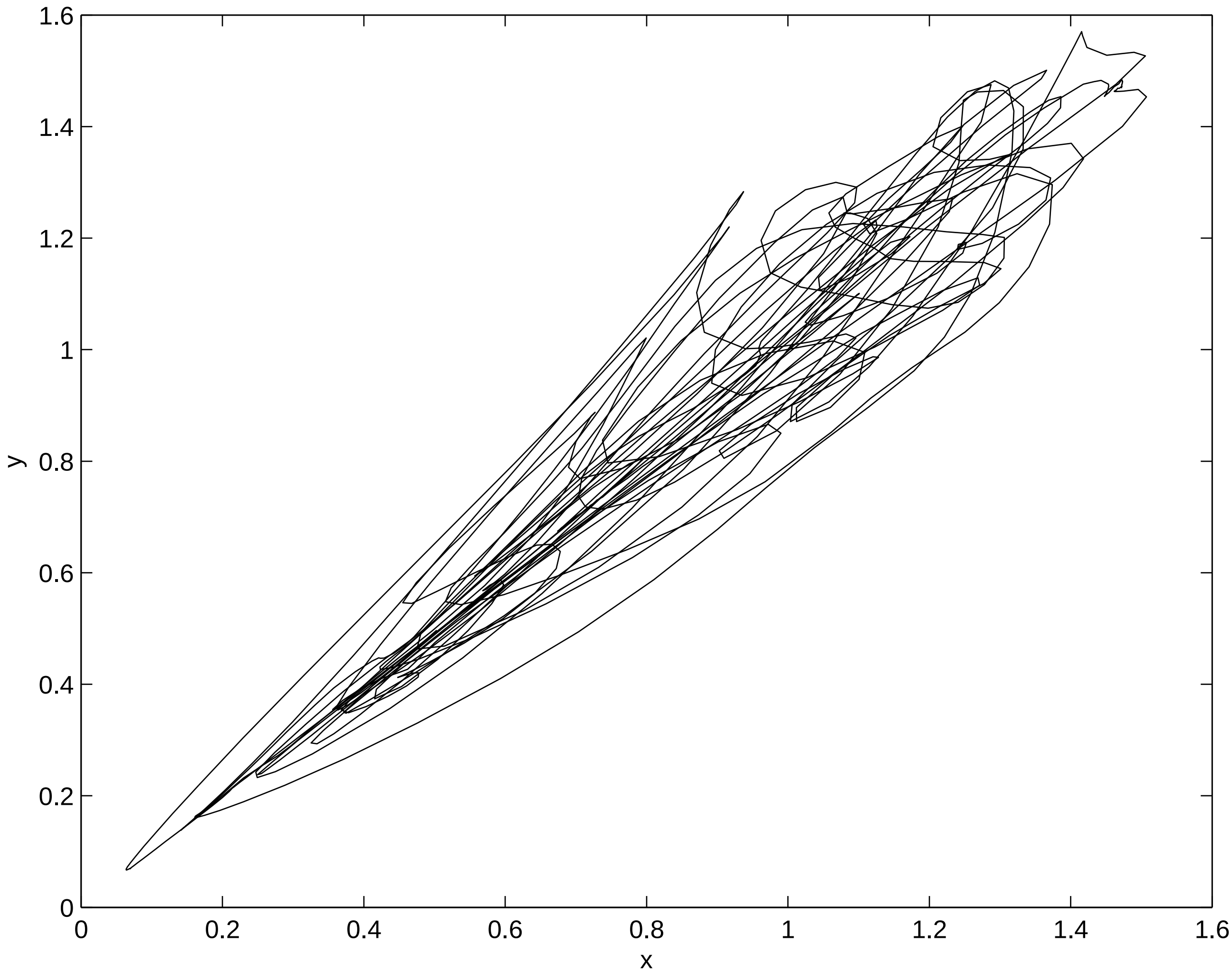

Fig.3.Shahverdiev PRL

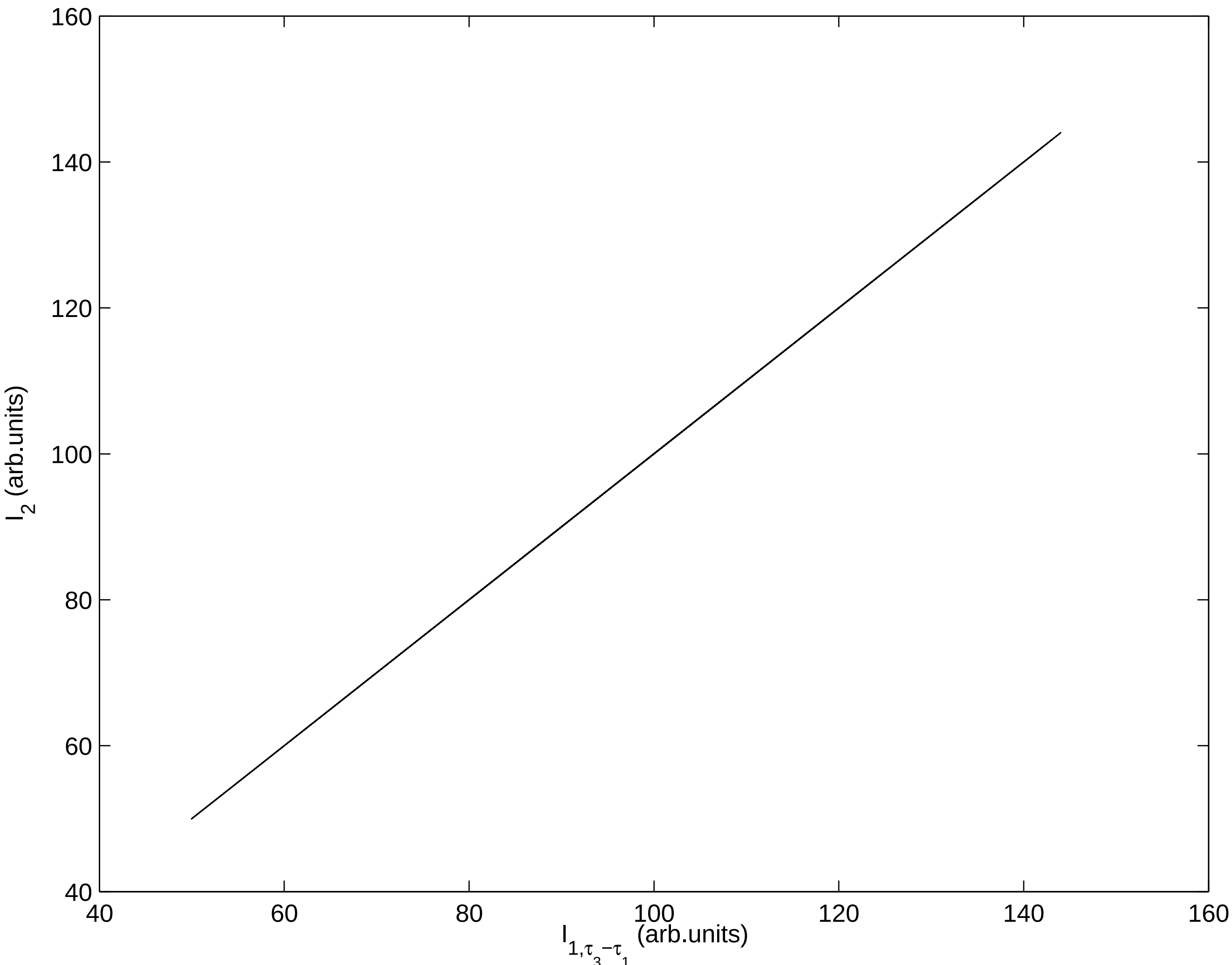

Fig.4.Shahverdiev PRL